Modifying the Welch Method to Estimate Power Spectral Percentiles

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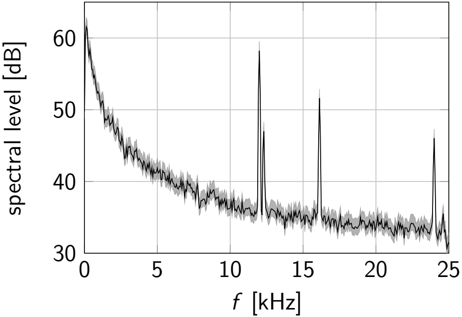
Research Effort

- **Common problem:** estimate power spectral density of noisy signals that are compromised by outliers
- **Common approach:** use Welch's method and replace the mean averaging by a median averaging



Schwock & Abadi, "Statistical Properties of a Modified Welch Method That Uses Sample Percentiles", IEEE ICASSP 2021, in review

Research Effort

- **Common problem:** estimate power spectral density of noisy signals that are compromised by outliers
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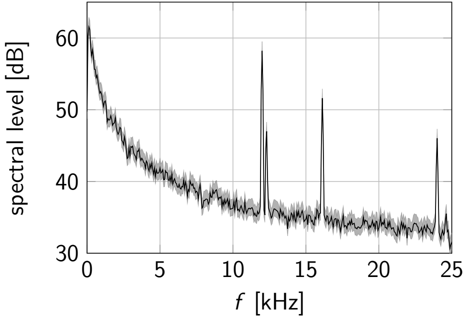
- **Our work:**
  - extend Welch median idea to a more general Welch percentile estimator
  - derive its statistical properties
  - use results to derive confidence intervals
  - analyze effect of outliers on percentile estimator



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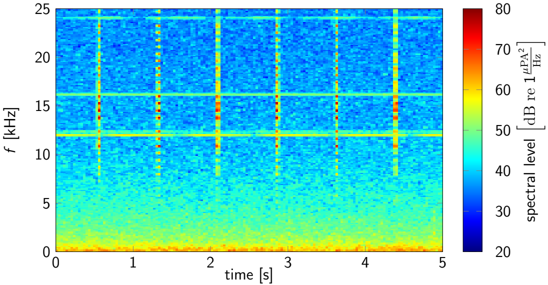
Standard Welch Method

Example: 5 s of ocean noise compromised by Acoustic Doppler Current Profiler (ADCP) pings



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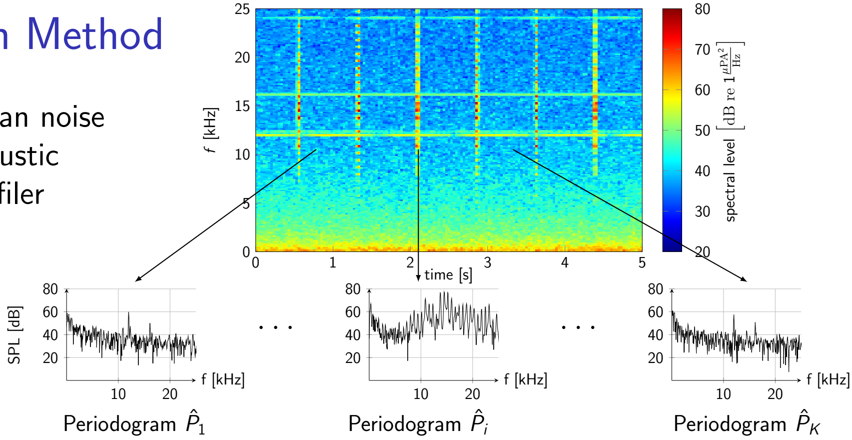

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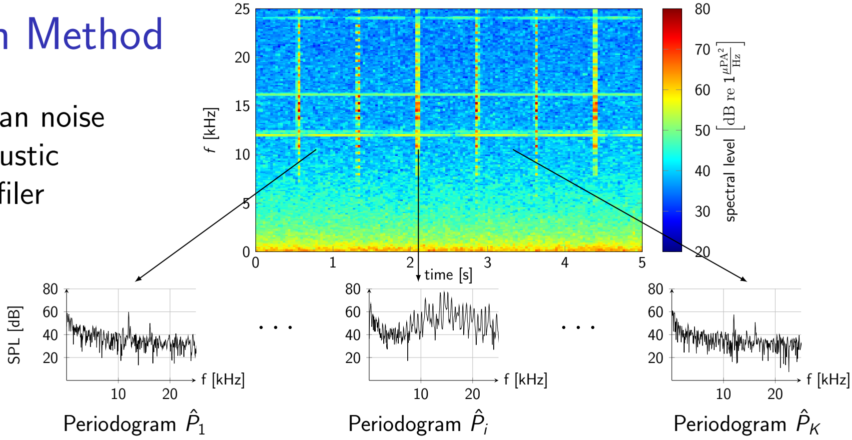

$$\hat{P}^{\text{Welch}}(f) = \frac{1}{K} \sum_{i=1}^{K} \hat{P}_i(f)$$

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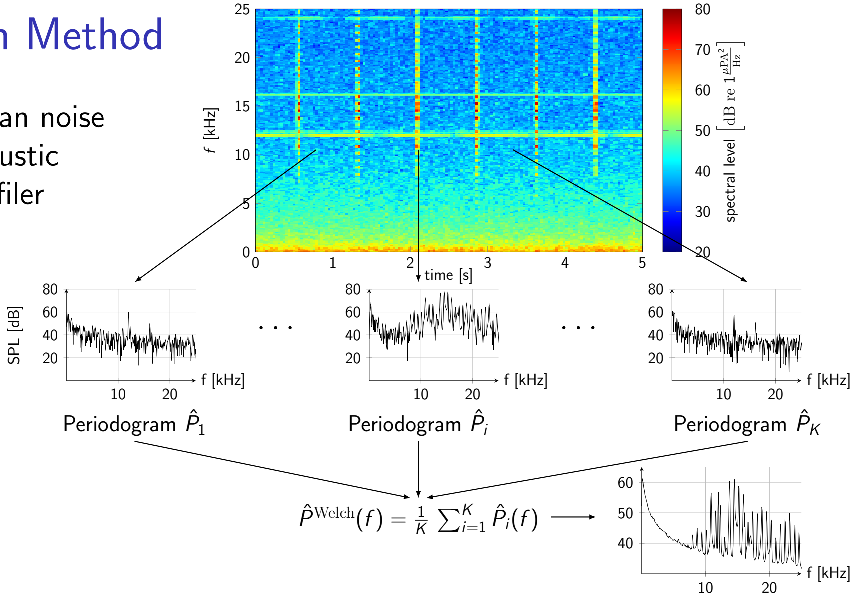

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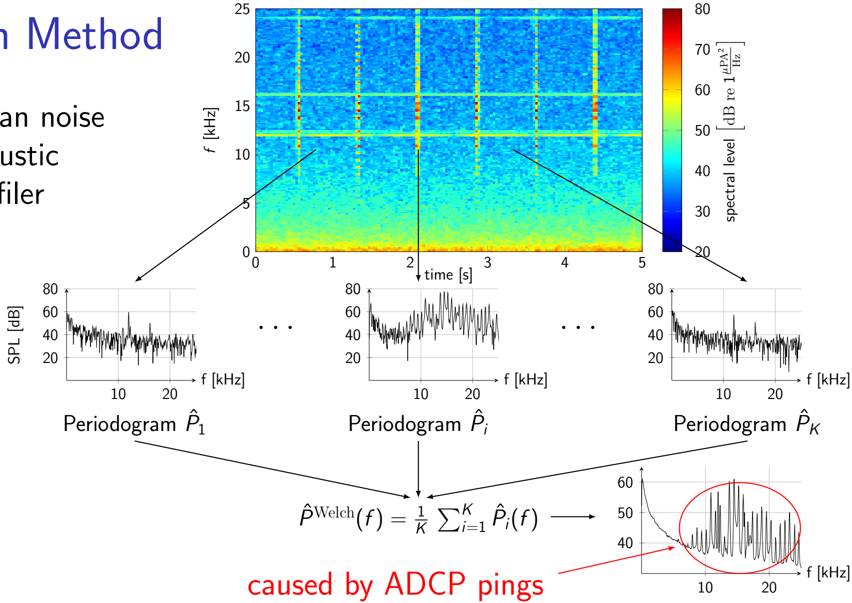

Welch Percentile Estimator

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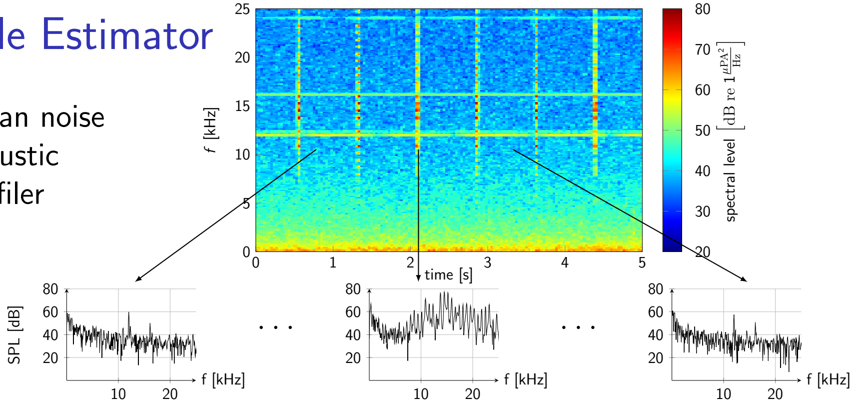

Periodogram $\hat{P}_1$ Periodogram $\hat{P}_i$ Periodogram $\hat{P}_K$

Welch Percentile Estimator

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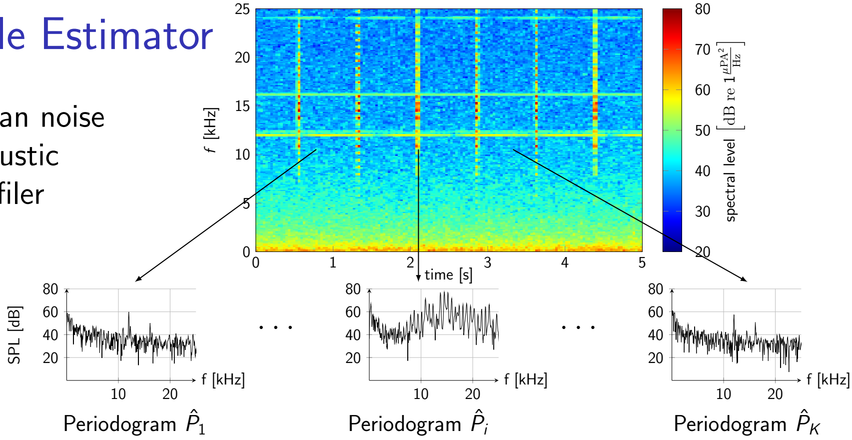

$$\hat{P}_q^{(\mathrm{WP})}(f) = \frac{1}{b} \cdot \underset{i}{\mathrm{quantile}_q} \left\{ \hat{P}_i(f) \right\}$$

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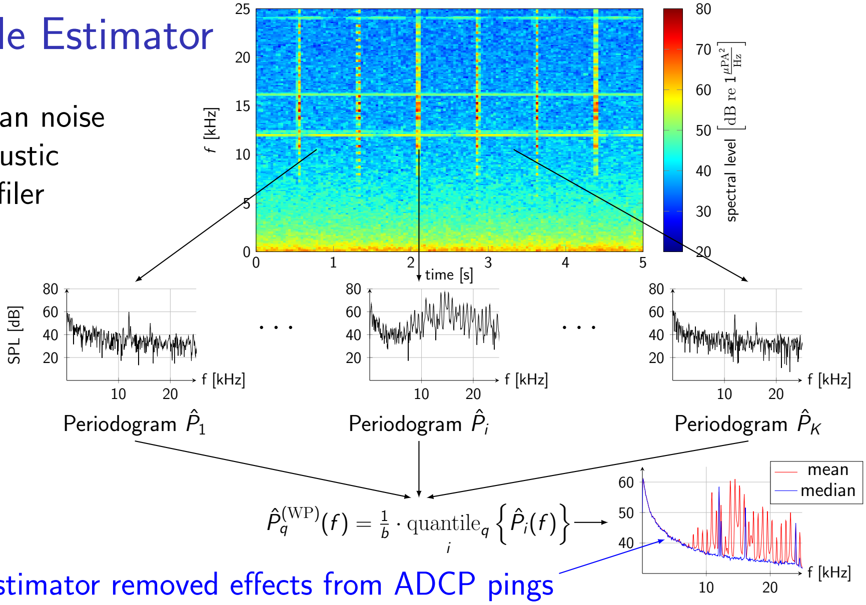

Statistical Properties of WP Estimator

- **Bias:**

$$b = \psi(K+2) - \psi(K(1-q)+1)$$

$$\psi(n) \approx \ln(n) - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6}$$

Variables

- b – bias
- K – number of periodograms
- q – q^{th} quantile
- ψ – digamma function
- ψ_1 – trigamma function
- P – true PSD

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- **Limiting Properties ($K \to \infty$):**

$$b = -\ln(1-q)$$

$$\mathrm{var}\left\{\hat{P}_q^{(\mathrm{WP})}\right\} = \left(\frac{P}{b}\right)^2 \cdot \frac{q}{K(1-q)}$$

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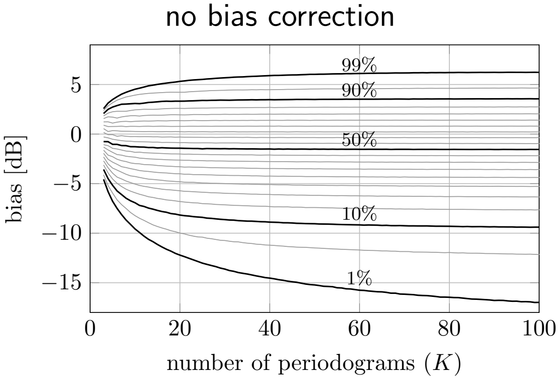
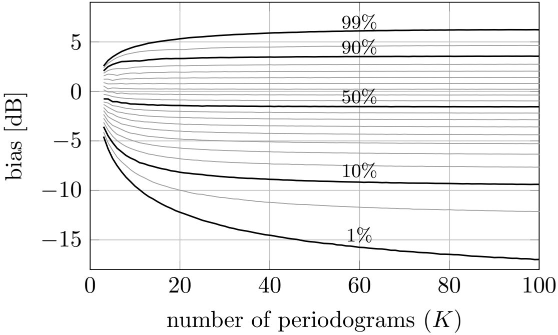
Simulation Results – Bias



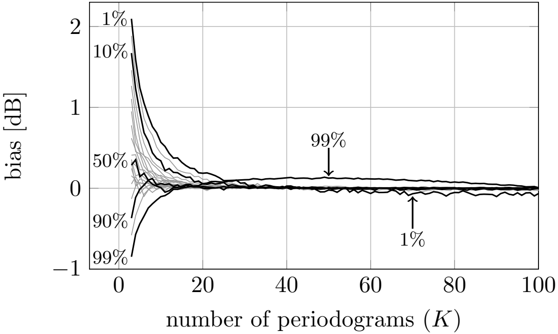
Simulation Results – Bias

no bias correction

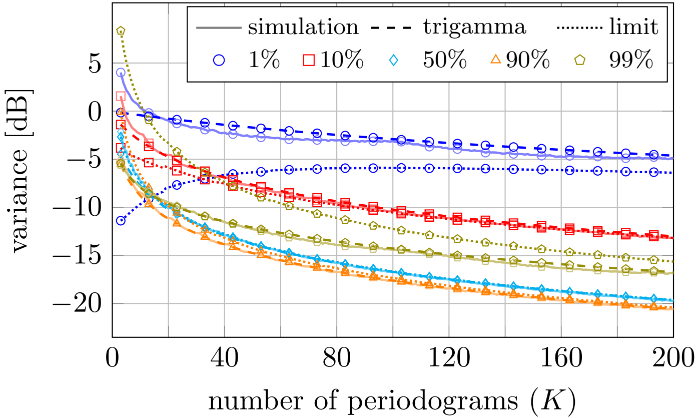


bias correction using

$$b = \psi(K+2) - \psi\left(K(1-q)+1\right)$$

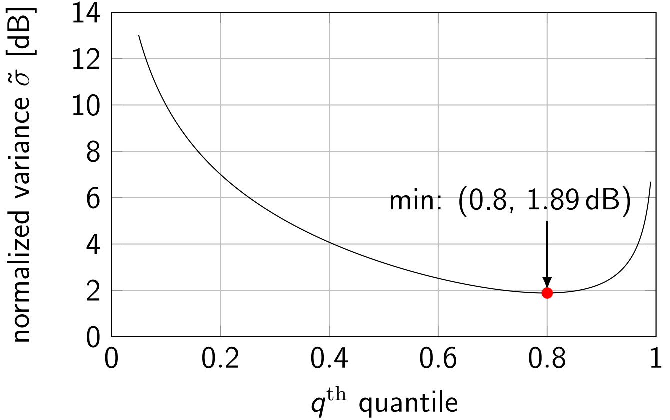


Simulation Results – Variance



- trigamma: $\mathrm{var} = \frac{P^2}{b^2}\left[\psi_1(K(1-q)+1) - \psi_1(K+2)\right]$
- limit: $\mathrm{var} = \left(\frac{P}{b}\right)^2 \cdot \frac{q}{K(1-q)}$

Lowest Variance Estimator



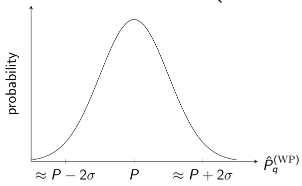
$$\tilde{\sigma} = \frac{\mathrm{var}\left\{\hat{P}_q^{(\mathrm{WP})}\right\}}{\mathrm{var}\left\{\hat{P}^{(\mathrm{Welch})}\right\}} = \frac{q}{b^2(1-q)},$$

with $b = -\ln(1-q)$

- $\mathrm{var}\left\{\hat{P}_{0.8}^{(\mathrm{WP})}\right\} = 1.54 \cdot \mathrm{var}\left\{\hat{P}^{(\mathrm{Welch})}\right\}$
- $\mathrm{var}\left\{\hat{P}_{0.5}^{(\mathrm{WP})}\right\} = 2.08 \cdot \mathrm{var}\left\{\hat{P}^{(\mathrm{Welch})}\right\}$

Confidence Intervals for Ocean Noise Signal

- $\hat{P}_q^{(\mathrm{WP})} \overset{\mathrm{d}}{=} P \cdot \mathcal{N}\left(1, \frac{q}{b^2 K(1-q)}\right)$



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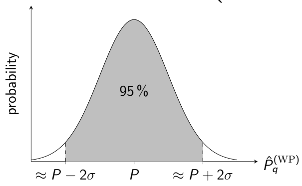

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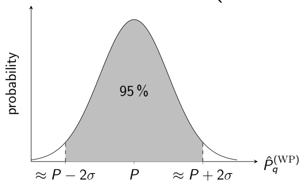
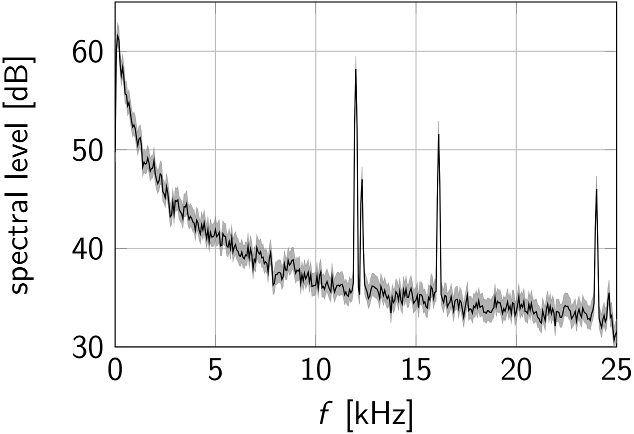


Parameters:

- $K = 124$
- $q = 0.8$
- Hann-window, 50 % overlap



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WP estimator performs well if percentage of outliers is small ($\leq 5\,\%$)

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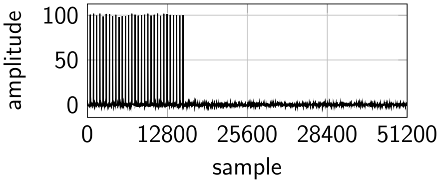
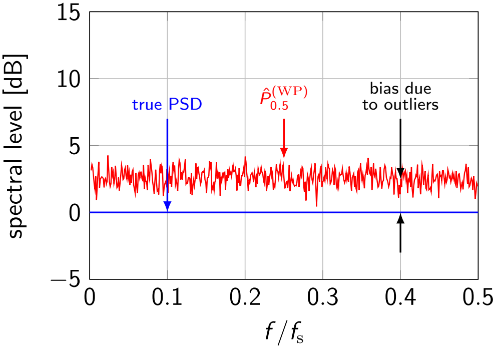
Large percentage of outliers:

- simulated white noise, $K = 100$
- $30\,\%$ of periodograms contain outliers





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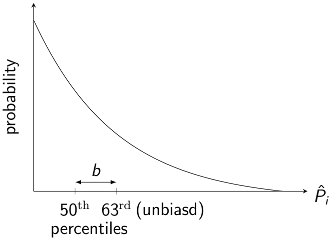

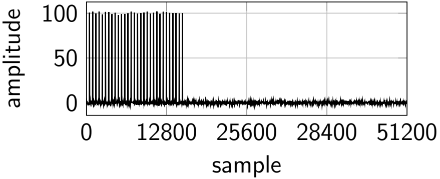

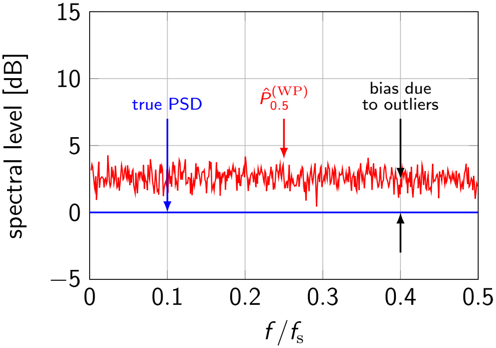

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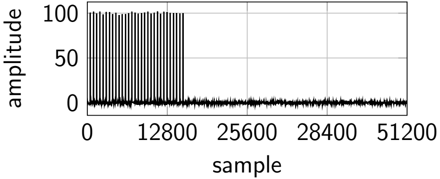

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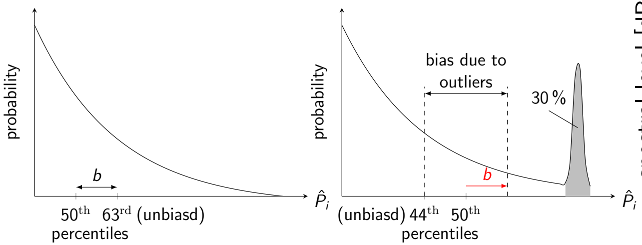
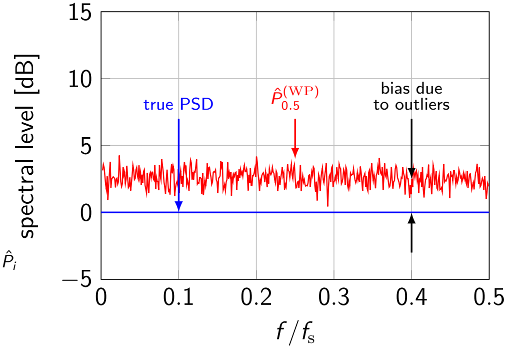
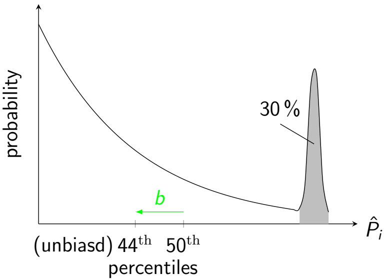




WP Estimator in Presence of Outliers (2)



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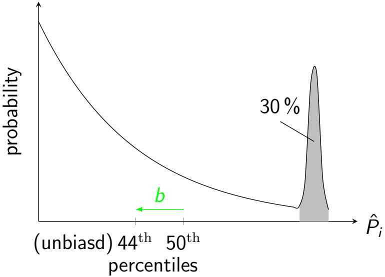

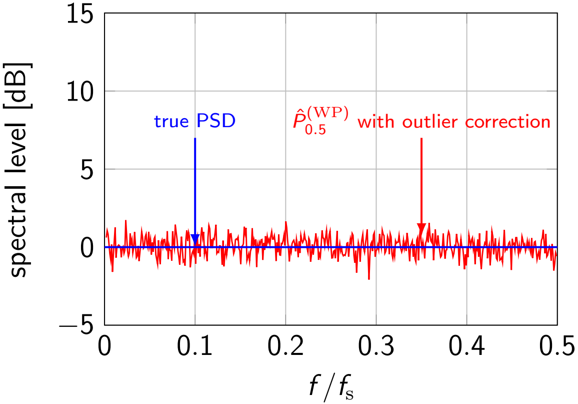

- correct for bias due to outliers by
  - $q \rightarrow \frac{q}{1-e}$, where e = percentage of outliers
  - $K \rightarrow K(1-e)$

Conclusions – Key Results

- simple expressions for bias, variance, and limiting distribution of the Welch percentile estimator can be derived from order statistics
- theoretical expressions show excellent agreement with simulations and can be used to derive confidence intervals
- the 80^{th} percentile estimator shows better variance properties than the commonly used 50^{th} percentile estimator
- bias and variance expressions can easily be adapted if percentage of outliers is large

Conclusions – Future Work and Applications

- **Future Work:**
  - improve theoretical results for small number of periodograms
  - adapt estimator if percentage of outliers is unknowns
- **Applications:**
  - estimate rain[1] and wind[2] noise spectral level in the northeast Pacific Ocean

[1] ASA Fall 2020 – 1pAOb4: Statistical Analysis and Modeling of Rain-generated Ocean Noise in the Northeast Pacific Ocean

[2] ASA Fall 2020 – 4aAOb3: Statistical Analysis and Modeling of Wind-generated Ocean Noise in the Northeast Pacific Ocean